Goldbach's Conjecture – A 280-Year-Old Unsolved Problem

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Abstract
In this paper, we present the content of Goldbach's conjecture about representing an even number as the sum of 2 primes and Chen's Theorem. The conjecture has not been proven, but in the middle of the 20th century, Chen Jingrun, a Chinese mathematician, proved half of the conjecture. The proof method is to use linear sieves to give an inequality in the number of ways of representing an even number as the sum of a prime and the other as a prime or as the product of 2 primes. Chen's entire proof is very long and uses a lot of advanced knowledge, so here we only restate the results' ideas and content.

Keywords: *Prime numbers, linear sieve.*

1. Introduction

In 1742, in a letter to Euler, Goldbach proposed a conjecture, later known as Goldbach's conjecture, predicting that every even number greater than 2 can be written as a sum of 2 prime numbers. For example:

$$8 = 3 + 5$$

$$10 = 3 + 7 = 5 + 5$$

$$20 = 13 + 7$$

With the aid of modern supercomputers, the conjecture has been proven to be true for all even numbers not exceeding 4×10^{18}. However, the conjecture is yet to be completely proven. A British company Faber promised to award $1,000,000 to the first person to solve the problem in two years, from 20th March 2000 to 20th March 2002. However, nobody won this reward, and now the problem remains unanswered for human beings.

In mathematics, number theory problems are regarded as the most difficult problems. Despite considerable effort, some problems look simple but have not been proven for years, and Goldbach's conjecture, named after mathematician Christian Goldbach (1690 – 1764) is one of them.

Christian Goldbach worked at the Imperial Academy of Sciences in Saint Petersburg. He was an excellent mathematician of the 18th century with his research on differential equations.

The problem that made him famous after 280 years was Goldbach's weak conjecture, known as the ternary Goldbach conjecture, asserting that every odd number greater than 7 can be written as the sum of 3 odd primes. For example, 35 = 19 + 13 + 3 or 77 = 53 + 13 + 11.

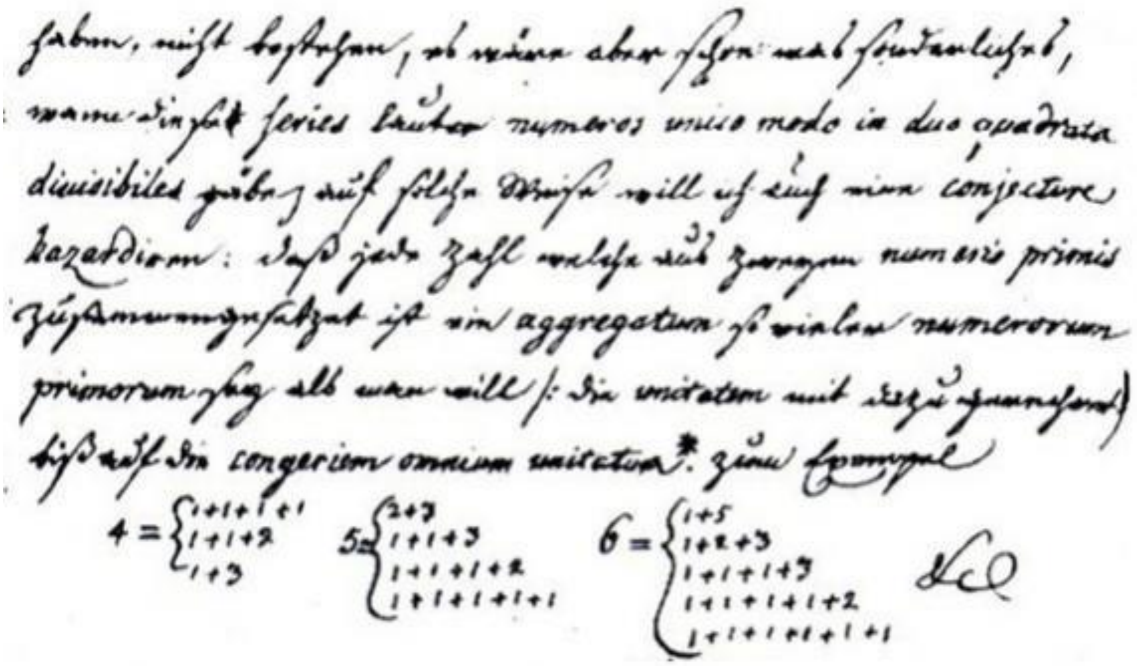

Figure 1. Goldbach's letter to Euler in 1742

A few years ago, Terence Tao has made great progress in proving Goldbach's weak conjecture by proving that every odd number can be written as a sum of no more than five prime numbers. Later, Harald Helfgott finished proving it. His solution is being verified, but we all hope it will help us prove Goldbach's weak conjecture. Like how it is named, if Goldbach's strong conjecture holds, the same is also true for the weak conjecture: to write an odd number as a sum of three different primes, we can subtract the number by 3 to have an even number, and apply Goldbach's strong conjecture for this even number.

However, Goldbach's conjecture remains unproven. It seems that we cannot use Terence Tao's proof of Goldbach's weak conjecture posted on his blog to prove the strong conjecture.

In the mid-20th century, Chen Jingrun proposed a theorem named after him – Chen's Theorem, which can be considered as a partial proof of the strong conjecture. After half a century, despite some effort, there haven't been any considerable breakthroughs. In this article, we want to give you Chen's unique, basic and simple to this problem. We hope you can find an innovative idea to prove Goldbach's conjecture!

Chen's Theorem states that *every even number that is great enough can be written as the sum of two prime numbers or the sum of a prime number and a semi-prime number (a number that can be expressed as the product of two prime numbers).* The theorem was first proposed by Chinese mathematician Chen Jingrun in 1966, [1] with detailed proof in 1973. [2] Chen's proof has been mostly simplified by P.M.Ross. Chen's theorem is a great breakthrough in proving Goldbach's conjecture, and a significant result of the sieve method.

2. Chen's Theorem

Now we will examine some basic ideas and talk about the main results that lead to the theorem. According to the theorem, every even number that is great enough can be expressed as the sum of two prime numbers or the sum of a prime number and a product of two prime numbers.

An integer that is a product of r distinct numbers is called an r-degree near-prime number, written as P_r. That means, Chen's theorem can be written as:

$$N = P + P_2$$

where N is a great enough even number.

Chen's Theorem.

Let $r(N)$ be the number of ways to express N in the form of:

$$N = p + n$$

where N is a great enough even number, p is an odd prime number, and n is the product of no more than two prime numbers. We have:

$$r(N) \gg \sigma(N)\frac{2N}{(\log N)^2}$$

where:

$$\sigma(N) = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p|N \\ p>2}}\frac{p-1}{p-2}.$$

Thus, if the theorem is proven, the number of ways to express $N = p + n$ is $\frac{N}{\log^2 N}$, which is a great enough number (greater than $\sqrt{N}$) , so there always exists a way to express N as shown above.

Now let's talk about the main idea of proving Chen's theorem.

Let N be an even number satisfying $N \geq 4^8$. We build a sequence of weighted numbers $w(n)$ for all positive numbers n. Let $z = N^{\frac{1}{8}}, y = N^{\frac{1}{3}}$

For $z \geq 4$, define $w(n)$ as follows:

$$\mathrm{w}(n) = 1 - \frac{1}{2}\sum_{\substack{z \leq q < y \\ q^k | n}} k - \frac{1}{2}\sum_{\substack{p_1 p_2 p_3 = n \\ z \leq p_1 < y \leq p_2 \leq p_3}} 1$$

It can be seen that $w(n) \leq 1$ for all positive integers n. The equality holds if and only if n has no prime divisors in the interval $[z, y)$.

In this article, let P be the set of prime numbers that are not divisors of N. We can see that $2 \notin P$ because N is an even number.

Let:

$$P(z) = \prod_{\substack{p \in P \\ p < z}} p$$

and let n be a positive integer satisfying:

$$n < N, GCD(n, N) = GCD(n, P(z)) = 1$$

Hence, n is only divisible to prime numbers not smaller than z and are not divisors of N. Assume that n can be written in the form of:

$$n = p_1 p_2 \cdots p_r p_{r+1} \cdots p_{r+s}$$

(n is the product of $r + s$ prime numbers)

so that:

$$z \le p_1 \le \ldots \le p_r < y \le p_{r+1} \le \ldots \le p_{r+s}$$

Hence, we have:

$$N^{\frac{s}{3}} = y^s \le p_{r+1} \cdots p_{r+s} \le n < N \to \frac{s}{3} < 1 \to s < 3$$

and therefore, $s \in \{0, 1, 2\}$.

Assume that $w(n) > 0$. We have:

$$\frac{1}{2} \sum_{\substack{q^k | n \\ z \le q < y}} k = \frac{1}{2}\left(\sum_{p_1^k | n} k + \sum_{p_2^k | n} k + \ldots \sum_{p_r^k | n} k \right) = \frac{1}{2}\left(1 + 1 + \ldots 1\right) = \frac{r}{2}$$

Hence:

$$\mathrm{w}\left(n\right) = 1 - \frac{1}{2} \sum_{\substack{z \le q < y \\ q^k | n}} k - \frac{1}{2} \sum_{\substack{p_1 p_2 p_3 = n \\ z \le p_1 < y \le p_2 \le p_3}} 1 = 1 - \frac{r}{2} - \frac{1}{2} \sum_{\substack{p_1 p_2 p_3 = n \\ z \le p_1 < y \le p_2 \le p_3}} 1 > 0 \to r \in \left\{0, 1\right\}$$

If $r = 1$ and $s = 2$, we have:

$$n = p_1 p_2 p_3, z \le p_1 < y \le p_2 \le p_3.$$

which leads to:

$$\mathrm{w}\left(n\right) = 1 - \frac{1}{2} \sum_{\substack{z \le q < y \\ q^k | n}} k - \frac{1}{2} \sum_{\substack{p_1 p_2 p_3 = n \\ z \le p_1 < y \le p_2 \le p_3}} 1 = 1 - \frac{1}{2} - \frac{1}{2} \sum_{\substack{p_1 p_2 p_3 = n \\ z \le p_1 < y \le p_2 \le p_3}} 1 = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

Thus, when $w(n) > 0$, these are the only cases that can happen: $r = 0$ and $s \in \{0; 1; 2\}$ or $r = 1$ and $s \in \{0; 1\}$. In both cases we have $r + s \le 2$. Hence, if $GCD(n, N) = GCD\big(n, P(z)\big) = 1$ and $w(n) > 0$, then n is the product of at most two prime numbers, or n is written in the form of: $n = p_1$ or $n = p_1 p_2$ where p_1 and p_2 are prime numbers not smaller than z and are not divisors of N.

Let:

$$A = \{N - p | p \le N, p \in P\}.$$

We know that A is a finite set of positive integers and

$$|A| = \pi(N) - \omega(N)$$

where $\pi(N)$ is the number of prime numbers not greater than N and $\omega(N)$ is the number of prime divisors of N.

If $n = N - p \in A$ and $GCD(n, N) = d > 1$, then p is divisible by d. This is not possible because p is a prime number that is not a divisor of n. Hence, $GCD(n, N) = 1$ for all $n \in A$.

We have:

$$r(N) \geq \sum_{\substack{N=p+n \\ n\in\{1,p_1,p_1p_2:p_1,p_2\geq z\}}} 1 \geq \sum_{\substack{n\in A \\ n\in\{1,p_1,p_1p_2:p_1,p_2\geq z\}}} 1 = \sum_{\substack{n\in A \\ (n,P(z))=1 \\ n\in\{1,p_1,p_1p_2:p_1,p_2\geq z\}}} 1 \geq \sum_{\substack{n\in A \\ (n,P(z))=1 \\ n\in\{1,p_1,p_1p_2:p_1,p_2\geq z\}}} \mathrm{w}(n)$$

$$\geq \sum_{\substack{n\in A \\ (n,P(z))=1}} \mathrm{w}(n) = \sum_{\substack{n\in A \\ (n,P(z))=1}} \left(1 - \frac{1}{2} \sum_{\substack{z\leq q<y \\ q^k|n}} k - \frac{1}{2} \sum_{\substack{p_1p_2p_3=n \\ z\leq p_1<y\leq p_2\leq p_3}} 1 \right)$$

$$= \sum_{\substack{n\in A \\ (n,P(z))=1}} 1 - \frac{1}{2} \sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{z\leq q<y \\ q^k|n}} k - \frac{1}{2} \sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{p_1p_2p_3=n \\ z\leq p_1<y\leq p_2\leq p_3}} 1$$

Let $A = (a(n))_{n=1}^{\infty}$ be the sequence of typical functions of A, which means if $n \in A$, $a(n) = 1$ and therefore:

$$\sum_{\substack{n\in A \\ (n,P(z))=1}} 1 = \sum_{(n,P(z))=1} a(n) = S(A, P, z)$$

We have the sum:

$$\sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{z\leq q<y \\ q^k|n}} k = \sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{z\leq q<y \\ q|n}} 1 + \sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{z\leq q<y \\ q^k|n \\ k\geq 2}} (k-1)$$

We have the sequence:

$$a_q(n) = \begin{cases} 1 \text{ } if \text{ n} \in \text{A and q}|\text{n} \\ 0 \text{ otherwise} \end{cases}$$

where q is a prime number.

Since $GCD(n, N) = 1$ $\forall n \in A$, we have: if $a_q(n) = 1$, $n \in A$ and q is a divisor of n. However, $GCD(q, N) = 1$, so q is not a divisor of N. Therefore, if $a_q(n) = 1$, then $q \in P$.

In that case:

$$\sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{z\le q<y \\ q|n}} 1 = \sum_{z\le q<y} \sum_{(n,P(z))=1} a_q(n)$$
$$= \sum_{z\le q<y} S(A_q, P, z)$$

We have:

$$\sum_{k=2}^{\infty} \frac{k-1}{q^k} = \sum_{k=2}^{\infty} (k-1) q^{-k} = \sum_{t=1}^{\infty} t q^{-t-1} = -\sum_{t=1}^{\infty} (-t) q^{-t-1} = -\frac{d}{dq}\left(\sum_{t=1}^{\infty} q^{-t}\right) = -\frac{d}{dq}\left(\frac{1}{q}+\frac{1}{q^2}+\dots\right) = -\frac{d}{dq}\left(\frac{1}{q}\cdot\frac{1}{1-\frac{1}{q}}\right)$$
$$= -\frac{d}{dq}\left(\frac{1}{q-1}\right) = \frac{1}{(q-1)^2}$$

In that case:

$$\sum_{\substack{n\in A \\ (n,P(z))=1}} \sum_{\substack{z\le q<y \\ q^k|n \\ k\ge 2}} (k-1) = \sum_{z\le q<y} \sum_{k=2}^{\infty} \sum_{\substack{n\in A \\ (n,P(z))=1 \\ q^k|n}} (k-1) \le \sum_{z\le q<y} \sum_{k=2}^{\infty} \sum_{\substack{n<N \\ q^k|n}} (k-1)$$
$$< \sum_{z\le q<y} \sum_{k=2}^{\infty} (k-1)\frac{N}{q^k} = N \sum_{z\le q<y} \sum_{k=2}^{\infty} \frac{k-1}{q^k} = N \sum_{z\le q<y} \frac{1}{(q-1)^2} < N \sum_{z\le q<y} \frac{1}{(q-1)^2}$$
$$< N \sum_{z\le q<y} \frac{1}{(q-2)(q-1)} = N \sum_{z\le q<y} \left(\frac{1}{q-2} - \frac{1}{q-1}\right) < N\left(\frac{1}{z-2} - \frac{1}{y-2}\right)$$
$$< \frac{N}{z-2} < \frac{2N}{z} = \frac{2N}{N^{\frac{1}{8}}} = 2N^{\frac{7}{8}}$$

With the third sum, let B be the set of all positive integers that can be written in the form of:

$$N - p_1 p_2 p_3$$

where the prime numbers p_1, p_2, p_3 satisfies:

$$\begin{cases} z \le p_1 < y \le p_2 \le p_3 \\ p_1 p_2 p_3 < N \\ GCD(p_1 p_2 p_3, N) = 1 \end{cases}$$

Let $(b(n))_{n=1}^{\infty}$ be the sequence of typical functions of set B. An element p of set B is a prime if and only if:

$$\begin{cases} p < N \\ N - p = p_1 p_2 p_3 \in A \\ z \le p_1 < y \le p_2 \le p_3 \end{cases}$$

Hence:

$$\sum_{\substack{n\in A\\(n,P(z))=1}}\ \sum_{\substack{p_1p_2p_3=n\\z\le p_1<y\le p_2\le p_3}} 1 = \sum_{\substack{p_1p_2p_3\in A\\z\le p_1<y\le p_2\le p_3}} 1 = \sum_{p\in B} 1 = \sum_{\substack{p\in B\\p<y}} 1 + \sum_{\substack{p\in B\\p\ge y}} 1 < y + \sum_{\substack{p\in B\\p\ge y}} 1 \le y + \sum_{\substack{n\in B\\(n,P(y))=1}} 1$$

$$= y + \sum_{(n,P(y))=1} b(n) = y + S(B,P,y) = N^{\frac{1}{3}} + S(B,P,y)$$

From these analyses, we get this theorem:

Theorem 1: We have:

$$r(N) > S(A,P,z) - \frac{1}{2}\sum_{z\le q<y} S(A_q,P,z) - \frac{1}{2}S(B,P,y) - 2N^{\frac{7}{8}} - N^{\frac{1}{3}}$$

Now we have to find a lower bound for $S(A,P,z)$ and an upper bound for $S(B,P,y)$ and $\sum_{z\le q<y} S(A_q,P,z)$.

3. Deriving to sieves to prove Chen's theorem

To apply the linear sieves in estimating the three sieve functions above, we choose a multiplicative function as follows:

$$g(d) = g_n(d) = \frac{1}{\varphi(d)}, n \ge 1$$

where $\phi(d)$ is the Euler's totient function. Since N is even, $2 \notin P$ (the set of prime numbers that are not divisors of N). Hence:

$$0 < g(p) = \frac{1}{p-1} < 1, \forall p \in P$$

It has been proven that there exists an $u_1(\epsilon)$ satisfying:

$$\begin{cases} \prod_{u\le p<z}\left(1-\frac{1}{p}\right)^{-1} < \left(1+\frac{\epsilon}{3}\right)\cdot\frac{\log z}{\log u} \\ u_1(\epsilon) \le u < z \end{cases}$$

Also, there exists an $u_2(\epsilon)$ satisfying:

$$\prod_{u_2(\varepsilon)\le p} \frac{(p-1)^2}{p(p-2)} = \prod_{u_2(\varepsilon)\le p}\left[1+\frac{1}{p(p-2)}\right] < 1+\frac{\varepsilon}{3}$$

Hence, we have: for all $u \ge u_0(\epsilon) = \max\{u_1(\epsilon), u_2(\epsilon)\}$:

$$\prod_{u\le p<z}\left(1-g(p)\right)^{-1}=\prod_{u\le p<z}\left(1-\frac{1}{p-1}\right)^{-1}=\prod_{u\le p<z}\frac{p-1}{p-2}=\prod_{u\le p<z}\frac{(p-1)^2}{p(p-2)}\prod_{u\le p<z}\frac{p}{p-1}$$

$$=\prod_{u\le p<z}\frac{(p-1)^2}{p(p-2)}\prod_{u\le p<z}\left(1-\frac{1}{p}\right)^{-1}<\left(1+\frac{\varepsilon}{3}\right)\left(1+\frac{\varepsilon}{3}\right)\frac{\log z}{\log u}<\left(1+\varepsilon\right)\frac{\log z}{\log u}$$

Let $Q(\epsilon)$ be the set of prime numbers p smaller than $u_0(\epsilon)$ and let $Q = P \cap Q(\epsilon)$. Let $\mathcal{Q}(\epsilon)$ be the product of prime numbers in $Q(\epsilon)$ and Q be the product of prime numbers in $\mathcal{Q}$. In that case, $\mathcal{Q}(\epsilon)$ only depends on ϵ and not on N. Therefore, $Q \le \mathcal{Q}(\epsilon) < \log N$ with great enough values of N (this is a result of the prime number theorem).

Theorem 2

Let N be a positive even number, and let

$$V(z)=\prod_{p|P(z)}\left(1-g(p)\right)=\prod_{\substack{p<z\\(p,N)=1}}\left(1-\frac{1}{p-1}\right)$$

In that case:

$$V(z)=\sigma(N)\frac{c^{-\gamma}}{\log z}\left(1+O\left(\frac{1}{\log N}\right)\right)$$

where:

$$\sigma(N)=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p>2\\p|N}}\frac{p-1}{p-2}$$

Theorem 3:

$$S(A,P,z)>\left(\frac{c^{\gamma}\log 3}{2}+O(\varepsilon)\right)\frac{N\cdot V(z)}{\log N}$$

Theorem 4:

$$\sum_{z\le q<y}S\left(A_q,P,z\right)<\left(\frac{c^{\gamma}\log 6}{2}+O(\varepsilon)\right)\frac{N\cdot V(z)}{\log N}$$

Theorem 5:

$$S\left(B,P,y\right) < \left(\frac{ce^{\gamma}}{2}+O\left(\varepsilon\right)\right)\frac{N\cdot V\left(z\right)}{\log N}+O\left(\frac{N}{\varepsilon\left(\log N\right)^{3}}\right)$$

where:

$$c=\int_{\frac{1}{8}}^{\frac{1}{3}}\frac{\log\left(2-3x\right)}{x\left(1-x\right)}dx=0.363..$$

Combining all theorems 4, 5 and 6, we have:

$$\begin{aligned}
r\left(N\right) &> S\left(A,P,z\right)-\frac{1}{2}\sum_{z\le q<y}S\left(A_{q},P,z\right)-\frac{1}{2}S\left(B,P,y\right)-2N^{\frac{7}{8}}-N^{\frac{1}{3}}\\
&=\left(\frac{e^{\gamma}\log 3}{2}+O\left(\varepsilon\right)\right)\frac{N\cdot V\left(z\right)}{\log N}-\frac{1}{2}\left(\frac{e^{\gamma}\log 6}{2}+O\left(\varepsilon\right)\right)\frac{N\cdot V\left(z\right)}{\log N}-\frac{1}{2}\left(\frac{ce^{\gamma}}{2}+O\left(\varepsilon\right)\right)\frac{N\cdot V\left(z\right)}{\log N}-2N^{\frac{7}{8}}-N^{\frac{1}{3}}\\
&>\frac{e^{\gamma}N\cdot V\left(z\right)}{4\log N}\left(2\log 3-\log 6-c-O\left(\varepsilon\right)\right)-O\left(\frac{N}{\varepsilon\left(\log N\right)^{3}}\right)\\
&=\frac{e^{\gamma}N\cdot V\left(z\right)}{4\log N}\left(0.042-O\left(\varepsilon\right)\right)-O\left(\frac{N}{\varepsilon\left(\log N\right)^{3}}\right)
\end{aligned}$$

We can choose:

$$\varepsilon=\frac{1}{1000000}\rightarrow O\left(\varepsilon\right)<0.042.$$

Therefore:

$$O\left(\frac{N}{\varepsilon\left(\log N\right)^{3}}\right)=O\left(\frac{N}{\left(\log N\right)^{3}}\right)$$

So we have:

$$r\left(N\right)>\frac{2N\sigma\left(N\right)}{\left(\log N\right)^{2}}\left(0.042-0.002\right)-O\left(\frac{N}{\varepsilon\left(\log N\right)^{3}}\right)$$

and therefore:

$$r\left(N\right)\gg\frac{2N\sigma\left(N\right)}{\left(\log N\right)^{2}}>0$$

The number of ways to express a great enough even number as a sum of a prime number and a product of no more than two prime factors is always larger than 0. In other words, we can always express a great enough even number as a sum of a prime number and a product of two prime numbers.

4. Result

In this paper, we have presented the idea of proving Chen's theorem: constructing a sieve described by a weight, which is not negative when the second term in Chen's representation has no more than two prime factors. Then Chen's number of representations is the sum of these weights and leads to linear sieves theorems 2, 3, 4, 5 that construct upper and lower bounds for these linear sieves. By using these sieves, we can conclude that every large enough even number can be written as a sum of two prime numbers or a sum of a prime number and a product of two prime numbers, which is a significant breakthrough in proving the Goldbach's conjecture.

5. Conclusion

The proofs of the relativity theorems mostly use analytic tools and estimators involving prime numbers and thus prove only half of the Goldbach conjecture. The Goldbach hypothesis stands tall like the top of the Everest mountain for 280 years but has not been conquerable. However, the statement of the hypothesis is so simple that it feels easy to solve. The next generations will follow in the footsteps of the previous ones; each generation will hold a milestone marking the passage that has been crossed. We can hope that with the development of modern mathematical tools, the Goldbach problem will be solved in the next few years.

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